A Critical Review on Buckling and Post-Buckling Analysis of Composite Structures

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Abstract

Main buckling and post-buckling analysis methods for composite structures are presented, along with the discussion on their potential challenges in handling the accuracy of prediction. Common buckling and post-buckling behaviors of composite structures are described, and the capabilities of corresponding analysis techniques are reviewed. Compared with the structures made of metals, the anisotropy of composite materials is one of the key difference which could be considered in constitutive relations for buckling analysis. Commonly used analytical buckling prediction methodologies are based on the energy principle for approximation; while post-buckling investigations usually require iteration algorithms such as the arc-length method etc. to trace the entire equilibrium path. Semi-analytical methods, such as finite strip method, are also effective algorithms to predict critical buckling loads, and they could significantly save computing resources compared with full numerical analysis models. Practical engineering structures are usually under elastically supported boundary conditions, and the buckling behaviors of composite panels under elastic constraints are also discussed. This brief review is intended to help the readers in identifying starting points for research in analysis of buckling and post-buckling behavior of composite structures and in designing composite structures against buckling.

Keywords

Composite Structures; Buckling; Post-Buckling; Structural Analysis; Shell Structures; Anisotropy

Introduction

Composites have been widely utilized in engineering structures since 1960s in the last century, and in some aspects they provide higher mechanical and chemical performances than their counterpart-metals, such as specific strength, corrosion resistance, etc. Therefore, the design and analysis methods of composite structures have been more and more widely investigated, especially for large scale engineering structures such as flight vehicles and marine structures. Both Boeing and Airbus have significantly increased the proportion of composites in their new generation of airliners, with reduced structural weight and the improved flight performance. For the requirement of vibration/noise control, the application of composites in helicopters is even more appealing. The increased proportion of composites in structure design has become one important index to gauge the advancement of a new type of aircraft.

Anisotropic composites provide more design freedom than conventional materials. However, their increased amount of design parameters incurs some difficulties in structural analysis. As shell structures, thin composite laminated plates might buckle before they reach the strength limit. Buckling of engineering structures may occur in a variety of forms, such as global or local deflections, and they might lead to the collapse of structures. Hence, avoiding buckling failure is an essential criterion in design of structural components. Although a lot of researchers have proved that structures still have the load carrying capacities in their post-buckling phase, the failure modes, such as fatigue characteristics, delamination status, etc., could not be accurately predicted. In aviation industry, buckling and excessive nonlinear deformation on some key parts of aircraft is not acceptable because of their effect on aerodynamic performance. Transverse shear flexibility effects of composite panels also become more pronounced due to their low ratios of transverse shear to in-plane moduli, and to attain exact analytical solutions is relatively difficult for anisotropic plates. Therefore, the instability of laminated structures is one of the most complex problems in structural mechanics, and a series of research work has been carried out in the past decades. A pre-research project POSICOSS (Improved POstbuckling SImulation for Design of Fibre COmposite Stiffened Fuselage Structures) started in

Europe since 2000 with main purpose to develop fast and reliable procedures for post-buckling analysis of fibre composite airframe structures, create experimental data base, and propose design guidelines. Another project COCOMAT (Improved MATerial Exploitation at Safe Design of COmposite Airframe Structures by Accurate Simulation of COllapse) was carried out to explore the accurate and reliable analysis methods of post-buckling up to collapse, and a series of simulation principles for buckling behaviors was investigated. In structures made of composites, the parts are usually thin-walled components. Composite orthotropic panels have similar geometric equations with isotropic materials, and the only difference is between their constitutive relations; thus, the classical computing methods, such as energy method, finite deference, finite element method, etc., are still suitable for buckling analysis of composite structures by recognizing orthotropic nature of the material in the stiffness formulae. Finite element analysis could provide high precision in predicting mechanical features of detailed parts, while analytical solutions provide higher computing efficiency for preliminary design. In this paper, some most concerned buckling features as well as their commonly used analysis methodologies of composite engineering structures are discussed that could provide a general overview for buckling prediction and design of composite structures.

Buckling Problem of Composite Structures

The equilibrium displacement of the structure carrying certain external load might become extremely large if the load reaches a critical level and increases further by a tiny quantity. The state at such a critical load level is generally called buckling, and the corresponding load is considered to be the critical buckling load. Buckling failure is usually caused by elastic instability. Due to thin-walled configurations, and plate/shell structures are more likely to buckle under compressive loads. External load usually represents nonlinear relationship with the structural deformation in buckling phase, and the gradient of load-deflection curve might be significantly decreased, representing the decrease of structural stiffness. In a complex structure, various buckling modes might occur according to the matching relationship between structural layout, stiffness parameters, dimensions of components etc. For an isotropic plate, the buckling waveshape is mainly related to its aspect ratio. In contrast, not only the dimensions (e.g., the aspect ratio), but also the anisotropic material property should be considered in composite panels since it could also affect the buckling deformation forms.

Theoretically, structures could still bear extra loads in their post-buckling phase. However, due to the bending and torque moment on the node and antinode lines of buckling waveshape, composite parts might debond from each other. Some other research studies found that the extremely large deformation might cause delamination between layers, and the initial buckling loads of delaminated composite panels are significantly lower than those in ideal situation. The crack propagation and fatigue prediction models in buckling conditions still need to be investigated, and the problem might be too complex to give consideration of all the failure states in a single analysis model. In engineering design, only some specific buckling characters are considered to be most effective on structural performance, such as the instability features under axial compressive or in-plane shearing load. On the other hand, the analyzed buckling character might be quite different from those of practical structures if some detailed features are not considered appropriately, and they are elaborated in the following sections.

Initial Geometrical Imperfection

Generally, structure components could not be in ideal shape due to the limitation of manufacture or assembling. Initial imperfections include geometrical imperfections, local ply-gaps, non-uniform applied end loads, and variations of boundary conditions etc., from which the initial geometrical imperfection is of most concern. Without the perturbation caused by imperfections, a finite element model with ideal shape could only represent the transmission of stress wave, and no buckling deformation will be predicted in nonlinear analysis. Geometrical imperfection could be in various forms, but it usually could not be exactly represented in a numerical model. Generally, the normalized buckling wave, obtained from eigenvalue buckling analysis, could be considered as an approximation of imperfection. Therefore, the introduction of the perturbation could be treated into two steps: eigenvalue buckling analysis is first processed to obtain the critical loads and deflection modes, and the deformation form is then defined on the structure by a small magnitude as geometric imperfections to process nonlinear post-buckling calculation. Linear buckling mode could provide accurate simulation of various geometrical imperfections, whereas different perturbation

forms might lead to different instability modes. Therefore, the analysis results still need to be corrected by experimental data. Some researchers discussed such simulation method of initial imperfections in ABAQUS environment, by employing the reduction integration element as well as the larger load increment before nonlinear response occurs, and nearly 75% of computational time was reduced without losing the accuracy of analysis.

Delamination-Driven Buckling

Impact of foreign objects or improper manufacture may cause delamination in composite panels, which is one of the most dominant damage forms of composite structures, and it may lead to significant reductions in their load-carrying capacity. Delamination crack may grow rapidly under compressive load, which is far different from metallic materials whose fatigue fracture occurs under extensional settings. Fibre microbuckling in delamination decreases the compressive strength of composite panels, and prediction models based on ideal assumptions may overestimate the critical load. Some researchers found that the compressive strength of composite panels is generally 30-40% lower than the tensile strength due to fibre microbuckling in delamination. Therefore, the prediction of delamination-driven buckling becomes an important branch of composite structural analysis.

A significant amount of research has been devoted to the buckling and post-buckling behavior of delaminated composite laminates, including analytical, numerical and experimental studies. Chai et al. investigated failure inside the laminated plates using a simple one-dimensional analytical model. The delamination area subdivides a square panel into four regions in their model, and for its simplicity, it was also developed into further steps by some other researchers. Suemasu studied the post-buckling behaviors of composite panels with multiple delaminations, in which an analytical method was formulated on the basis of Rayleigh-Ritz energy approximation technique, and the contact between delaminated surfaces is considered by the constrained points in the delaminated area. Zhang and Wang discussed a layer-wise B-spline finite strip method with consideration of delamination surfaces to study the post-buckling behavior of debonded composite laminates under in-plane loads, and their result showed that initial buckling load of delaminated composite laminates is often much lower than the intact laminate buckling load.

Delamination usually involves contact leading to nonlinear post-buckling problems, and analytical models might not give consideration of all the factors simultaneously. Numerical simulations and experiments are needed to provide further investigation on the phenomenon. Virtual crack closure technique (VCCT) and cohesive interface element model are commonly used in the finite element analysis of delamination features. The two methods employ similar principle, and they simulate the crack propagation by the energy release rate criterion. The principle of VCCT is based on classical fracture mechanics, which investigates the behavior of crack propagation when an initial crack is designated. While cohesive element is a numerical model based on damage mechanics, in which the stiffnesses of the element decrease when passing a critical stress, until a complete failure is achieved, and then the bonding element is eliminated to simulate the propagation of a fracture. Therefore, the entire process of crack initiation to propagation could be analyzed by cohesive models. Delamination-driven buckling is a highly nonlinear and discontinuous problem, and the potential fracture area should be densely meshed to obtain a converged result. Therefore, the computational efficiency of the implicit algorithm based on arc-length method still needs to be improved.

Thermomechanical Buckling

At high temperatures, composite plates are found to buckle without the application of mechanical loads. Hence, buckling characteristics of composite panels under thermal loads have to be understood. Besides finite element methods, thermomechanical buckling could also be predicted using approximation theories such as energy principles, etc. Shen developed a higher-order shear deformation plate theory to study the thermal buckling of a simply supported composite laminated plate subjected to uniform and non-uniform tent-like temperature loading. A mixed Galerkin-perturbation technique is used to determine thermal buckling loads and post-buckling equilibrium paths. Various design parameters have been taken into account to investigate their effect on thermal buckling behavior. The results show that the characteristics of thermal post-buckling are significantly influenced by the transverse shear deformation, thermal load ratio, plate aspect ratio, fiber orientation and initial geometrical imperfection, whereas the total number of plies has rather less effect. Different computational methods used for high temperature composite panels were reviewed by Noor and Burton. Topal et al.

presented optimal design of laminated plates subjected to uniformly distributed temperature. The modified feasible direction method was employed to maximize the critical temperature and the fibre orientation was considered as design variable, in which the buckling displacement field is expressed according to the first-order shear deformation theory. Based on the similar algorithm, Ahmed et al. studied the thermomechanical buckling of composite panels with circular cutouts, and the effects of the variations in design parameters, such as hole diameter, stacking sequence, etc., were investigated. Other researchers evaluated the thermal post-buckling behavior of graphite/epoxy laminated plates in various boundary conditions using finite element method, in which the nonlinear governing equations were solved as a sequence of linear eigenvalue problems to trace the thermal post-buckling path. For simplicity, the material properties were assumed to be independent of temperature in many studies.

Methods for Buckling Analysis of Composite Structures

The main purpose of buckling analysis is to predict the critical load and buckling deformation, from which the buckling resistance and carrying capacity of the structure can be understood. For an isotropic panel, the critical buckling load usually depends on the aspect ratio and the stiffness. In contrast, analysis of composite panels must also consider the effect of anisotropy, which makes the problem more complex. In engineering field, the stacking sequence of the laminated plate is generally designed to be symmetric and balanced to avoid unpredictable warp deflections. The critical load of a balanced symmetric or orthotropic laminated plate could be estimated by:

$$N_{cort}=\frac{2\pi^2 D_{22}}{b^2}\left[\alpha\sqrt{\frac{D_{11}}{D_{22}}}+\beta\frac{\left(D_{12}+2D_{66}\right)}{D_{22}}\right] \qquad (1)$$

where D_{ij} (i,j=1,2,6) represents the bending stiffness coefficients of the laminate, and α and β are the correction factors determined by the boundary support conditions of the panel. Experimental data could be employed to adjust the correction factors in empirical formulas. Generally, experiments should be conducted to obtain the crucial data for typical structures, which could be summarized into design curves, correction factors, formulas etc. These empirical data provide reliable and simplified models for specific problems. For more complex structures, finite element model or some other approximation methods are considered to predict the buckling load. Athiannan compared the result of Timoshenko's classical formula with experimental data in the shear buckling problem of a cylindrical shell, and the result showed an acceptable accuracy. The buckling analysis algorithms involve complex elastic/plastic theories, and even for a single square panel, high order partial differential equations need to be solved to determine the critical load. The most commonly used buckling analysis method is based on classical linear buckling theories, which assume that the structure might collapse as soon as the external load reaches its critical value. Linear buckling theories were oriented from Euler's investigation on the carrying ability of columns, and they were then improved by Timoshenko in shell structures. Linear buckling theories were preliminary based on the assumption of small deflection and linear elasticity without considering the effect of initial imperfection or deformation. While nonlinear buckling theories were developed to simulate the post-buckling behaviors with higher precision, which could trace the load-deformation relationship by the iteration methods. However, the iteration-based nonlinear algorithm requires much more computation. With the development of improved computer performance and finite element methods, nonlinear algorithms have become common in engineering analysis.

Numerical Analysis Based on Finite Element Models

The state-of-art in buckling analysis of laminated plates was discussed by some experts such as Leissa and Chia et al. Finite element method (FEM) has become one of the most widely used and dependable tools, which can provide meaningful and accurate results regardless of the complexities in geometry, material properties, boundary conditions and loading. While the usage of FEM is not without problems, and shear locking might be one of the challenges, which usually occurs in lower-order quadrilateral element based on the first-order shear deformation theory. Shear locking elements might exhibit high stiffening behavior leading to higher buckling loads. Many techniques have been tried over the years to overcome this shear locking phenomenon. Singh et al. developed an accurate four-node plate element to deal with shear locking problem in buckling analysis. Their element employs coupled displacement field which is derived using static equations of equilibrium. For buckling analysis, finite element theory could be mainly classified into two categories: one is basically the

linear method which determines buckling load by eigenvalue extraction, and the other is the incremental arc-length method in combination with Newton-Raphson iteration to trace the load direction and path, which could handle nonlinear problems.

In linear buckling analysis, the critical load and buckling mode are obtained by solving the eigenvalue of stiffness matrix in linear system. Linear buckling problem of finite element analysis could be expressed as [1]:

$$\left[[K]-\lambda\left[K_g\right]\right]\{\delta\}=0 \tag{2}$$

where $[K]$ and $[K_g]$ are the global elastic stiffness matrix and geometric stiffness matrix, respectively. λ and $\{\delta\}$ represent the eigenvalue and displacement eigenvector, respectively. The minimum value of λ corresponds to the normalized initial buckling load, which is usually called the buckling factor, and the corresponding eigenvector gives the mode shape. One of the most commonly used eigenvalue extraction algorithm is Lanczos method. This method could reduce the number of non-zero elements in the coefficient matrix of eigenvalue equation by Lanczos vector, and it extracts multiple orders of eigenvalues with high efficiency. Eigenvalue buckling analysis could not trace the actual deformation. Without considering the effect of dynamic deflection and initial imperfection, the linear buckling model might slightly over-estimate the carrying capacity of structures. However, they usually provide high computing efficiency and are widely used for the prediction of overall structural performance.

Post-buckling characters usually represent extremely large deflections and need non-linear algorithm for prediction. In non-linear incremental approach, the displacement increment iterates along the load equilibrium path, which is usually called the arc-length method. The solution is searched using incremental and iterative algorithm, where the loads are applied by a serial of steps. For each load level an iterative process is performed in order to reduce the errors transferred to the next load level. Thus the entire load-deformation path could be traced. Lee proposed a modified explicit arc-length method by combining dynamic relaxation with kinetic damping technique, which does not require the computation of tangent stiffness matrix to search the equilibrium path. For some highly non-linear structural instability problems, such as buckling mode-switching, either the classical arc-length method might become less effective, since the problem could be considered as a transient dynamic event, or the continuation methods might lead to difficulties when simulating this phenomena. Thus, the explicit dynamic analysis might be needed to depict the highly non-linear transient responses. However, the inefficient iterations for the convergence might be the main limitation of the explicit algorithm.

Analytical Solutions for Buckling Problems

Numerical finite element models could provide precise prediction of mechanical features but requires large amount of computational resources, which might be more suitable for detailed analysis rather than overall demonstration of structural performance. Instability behaviors of composite structures are usually caused by in-plane compressive or shearing loads, and analytical methods for some simplified structures at component level are still widely investigated for their acceptable accuracy and high calculation efficiency. Some researchers completed the optimization of a rocket launch structure by analytical approximation, and the result proved that the stiffened shell structure could save more than 40% of structural weight compared to that of unsitffened thick panel under buckling constraints. Another investigation optimized the layout of a composite wing structure considering various buckling modes, and they combined empirical equation with surrogate model to predict local buckling features for composite wing panels. Approximation models for buckling problems usually employ energy principle or semi-analytical approaches to determine the critical buckling load of structures, of which the Rayleigh-Ritz method, Kantorovich method, finite strip method, etc. are commonly used. Most analytical methods are governed by orthotropic or anisotropic plate buckling theory for laminates which are symmetrically stacked with respect to the plate mid-plane. Unsymmetrical laminates require a more complicated theory with bending-stretching coupling.

1) Rayleigh-Ritz Method

Based on energy variational theory to solve the buckling loads, energy principle is one of the most commonly used theories for the approximation of buckling analysis. The crucial step of this method is to select an appropriate displacement shape function in order to properly describe the deflection of the plate in its buckled state and at the same time to satisfy the boundary conditions. The critical buckling load is then obtained by solving characteristic equation according to the minimum

potential energy principle. Rayleigh-Ritz energy method is proved to be effective in analyzing the global buckling load of orthotropic plates. According to its principle, the total potential energy should be a minimum to make equilibrium stable, and the corresponding forces are critical buckling loads. The total energy *Π* of the system could be expressed by the algebraic superimposition of the potential energy *V* introduced by the external forces and the strain energy *U*, respectively, which is represented in the mathematical form as:

$$\Pi = U(\omega) + V(\omega) \tag{3}$$

Based on the theory of elasticity, *U* and *V* could be expressed by the equations including displacement shape function $\omega(x,y)$ as:

$$\begin{cases} U = \frac{1}{2}\iint [D_{11}(\frac{\partial^2\omega}{\partial x^2})^2 + 2D_{12}\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2}D_{22}(\frac{\partial^2\omega}{\partial y^2})^2 \\ +4(D_{16}\frac{\partial\omega^2}{\partial x^2} + D_{26}\frac{\partial^2\omega}{\partial y^2})\frac{\partial^2\omega}{\partial x\partial y} + 4D_{66}(\frac{\partial^2\omega}{\partial x\partial y})^2]dxdy \\ V = \frac{1}{2}\iint [N_x\left(\frac{\partial\omega}{\partial x}\right)^2 + N_y\left(\frac{\partial\omega}{\partial y}\right)^2 \\ +2N_{xy}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}]dxdy \\ \omega(x,y) = \sum_{i=1}^{n} C_i\varphi_i(x,y) \end{cases} \tag{4}$$

The shape function usually consists of polynomials with the undetermined coefficients C_i. The minimum potential energy could be obtained by finding the first derivative of the total potential energy with respect to these unknown constants and equating it to zero:

$$\partial\prod/\partial C_i = 0 \tag{5}$$

This expression could be considered as a system of linear equations with respect to constant C_i, and it results in an eigenvalue problem. By solving the minimum eigenvalue, the critical buckling load could be obtained.

In most cases, the Ritz method could derive an explicit result for buckling load. Therefore, the prediction of critical load does not require much computing resources, which is one of the main advantages of the analytical methods. Eirik et al. developed an analytical model for buckling analysis of stiffened panels, where the computational model was prompted by considering the stiffened panel as a plate with anisotropic stiffness coefficients, and the Ritz energy principle is utilized to derive the equilibrium equation. Their results have demonstrated a satisfactory agreement with ABAQUS computed data.

2) *Galerkin Method*

Galerkin method is another effective algorithm for solving differential equations, and it could also be used to establish an eigenvalue problem for linear buckling analysis. Galerkin equation system could be given by:

$$\iint L(\omega)\varphi_i(x,y)dxdy = 0 \tag{6}$$

where $L(\omega)$ is the governing differential equation of the panel, and the shape function could has a similar form as that in equation (4). Equation (6) could also be considered as a linear equation system of C_i. For a nonzero solution, the coefficient determinant should be equal to zero. Thus, an eigenvalue problem could be obtained for buckling load, which is similar with the principle of the Ritz method. Fiorenzo and Erasmo investigated the accuracy of plate theories for buckling and vibration analysis, and the results were obtained based on both the Ritz and Galerkin methods. When the boundary terms in the governing equations are neglected, the two methodologies lead to the same results. Jaberzadeh formulated the mathematical relationship between buckling capacity and the plate aspect ratio, as well as the boundary rotational stiffness using Galerkin method. The contribution of rotational constraint stiffness is converted into part of the strain energy, which could be considered as an effective approximation for the elastically-constrained boundaries.

Based on the first-order shear deformation theory, a modified meshless Galerkin method is used for solving buckling problems. The meshless method discretizes the structure by a series of distributive nodes. The approximative solution could be derived by interpolation among these nodes. The meshless Gelerkin method could be used to establish the stiffness equation, and then the governing equations of buckling behaviour could be derived by combining the stiffness matrix and stress matrix. Since meshless method does not rely on finite elements, some problems caused by mesh distorsion can be avoided by this approximation model

3) *Finite Strip Method*

In the analysis of shell structures, the finite strip method (FSM) could be considered as an efficient method to predict buckling loads, which is under

wide investigations. The major two variants of FSM are semi-analytical FSM (S-A FSM) and the spline FSM (SFSM). The basic difference between them lies in the longitudinal representation of the displacements. In the semi-analytical FSM, the representation of a displacement quantity in the longitudinal direction is by a series of smooth, continuous analytical functions which run the whole length of the structure. While in the spline FSM or SFSM, a displacement function is represented in the longitudinal direction by a series of local, polynomial spline functions. Compared with the former one, the spline FSM is a more versatile procedure, with the advantage that the longitudinal expression remains the same whenever the end condition change. The basic philosophy of FSM is to discretize the structures into longitudinal strips and interpolate the behavior in the axial direction by different functions, which could be considered as some kind of simplified finite element method in which a special element called strip is used. Since discretization is only applied in one direction (i.e., in the transverse or circumferential direction) and the other direction (i.e., the longitudinal direction) of the panel employs analytical functions, a finite strip model has much fewer degrees of freedom than the finite element model. Therefore, it could be more computationally efficient.

A great deal of research has been carried out to investigate the applicability of the finite strip method. Subsequently, many useful extensions have been developed. Wang investigated analysis capability of predicting the bucking stresses of prismatic shell structures using B-spline FSM. Dawe et al. studied the response of plates using the semi-analytical finite strip method for plates with symmetric initial imperfection, in which the geometric functions were represented by Fourier series. Ovesy described both spline and S-A FSM for predicting the post-buckling response of composite laminated plates with initial imperfections, in which the geometric nonlinearity was introduced in the strain-displacement equations in the manner of von Karman assumptions, and the formulations of the finite strip methods were based on the concept of the principle of the minimum potential energy. Their study of the results revealed that the buckling response of composite panels is strongly influenced by the form and magnitude of initial imperfections. Some other researchers studied the post-buckling analysis of composite laminated plates subjected to progressive uniform end shortening, using the spline finite strip method with both classical plate theory and shear deformation plate theories. Ovesy and Assaee developed a non-linear multi-term finite strip method for the post-buckling analysis of thin-walled symmetric cross-ply laminated plates under uniform end shortening. Their method was based on solving von Karman's compatibility equation to obtain mid-plane stresses and displacements, and then, by invoking the principle of the minimum potential energy, the equilibrium equations of finite strips were derived. The results were discussed in detail and compared with those obtained from the finite element analysis, which provided confidence in the validity and capability of the developed finite strip model in handling the post-buckling problem.

Finite strip method invokes analytical shape functions and discretization model in longitudinal and transverse directions, respectively, and it combines the merits of both analytical and numerical methods and provides satisfactory accuracy as well as computational efficiency, especially in the buckling problems of plate/shell structures. The determination of shape functions for the finite strip is a main issue of FSM. An unreasonable deflection shape function might cause difficulties in post-buckling analysis and at the same time could affect accuracy of solutions. To provide higher accuracy for complex engineering structures is another challenge of such a semi-analytical method, which is based on simplified mathematical models. Therefore, further investigations need to be processed to improve the practicability of this method.

4) *Kantorovich Method*

Some researchers developed another algorithm named Kantorovich method based on the variational principle, which is another semi-analytical method requiring iteration to reduce a set of governing partial differential equations into the governing ordinary differential equations (ODE), and the converged solution could be considered as the critical load. Computation examples show that the convergence of solutions of Kantorovich method is very fast. The iterative equations can be derived either by using the Galerkin equation or other approximations based on the principle of minimum total potential energy

for eigenvalues and eigenvectors. The iteration procedure is repeated until the eigenvalue converges to a specific value of critical buckling load. Since the displacement functions could be automatically forced to satisfy the boundary conditions during the iterations, an advantage of this method is that the initial assumed displacement function in the first iteration could be arbitrarily selected regardless of the type of boundary conditions.

5) Approximation of Elastically Restrained Boundaries

Approximate analytical methods usually make some significant simplification for composite plate/shell structures. Theoretical analysis of composite structures at component level usually considers that the composite panels are pinned or rigidly supported by other adjacent structures, which might be difficult to achieve in real practice. Boundary supports in real structures usually provide finite value of rigidity, which is in contrast to the idealized simply or clamped supported boundary conditions. Design guidelines for restraning capacity might lead to conservative result if they are based on the simply supported boundary conditions. More accurate buckling analysis could lead to a lighter weight design; therefore, the elasticity of boundary constraint should be taken into account.

The methods referring to buckling analyses with elastic restraints are applied mainly to flat plates, and then extended into stiffened orthotropic panels, which is a typical structure form in engineering field. Most of the analytical buckling approaches are based on the energy principles, where the contribution of boundary constraint elasticity is considered as one additional term in the expression of structural deformation energy. Stamatelos et al. considered a segment of stiffened panel and replaced the stiffeners by equivalent transverse and rotational springs of varying stiffnesses, which act as the elastic edge supports. The buckling analysis was processed based on the Ritz energy method, and a comparison of their methodology results to respective finite element results demonstrated a satisfactory agreement. Bisagni developed an analytical formulation for local buckling and post-buckling analysis of isotropic and laminated stiffened plates. The structure was evaluated considering the skin of the stiffened plate located between two stiffeners. The restraint to plate edge rotation provided by stiffeners was taken into account through a Saint-Venant torsion bar. Paik and Thayamballi developed simple design formulations for buckling strength as a function of the torsional rigidity of support members, and the result demonstrated that, as the stiffness of stiffener becomes larger, its enhancement effect increases nonlinearly from the initial value of a simply supported boundary to that of a clamped boundary. By applying Ritz method to an eigenvalue problem, Qiao and Zou derived the explicit solution of buckling load for orthotropic panels carrying non-uniform compression and elastically restrained at the unloaded edges. By weighting the buckling shape functions corresponding to simply supported and clamped boundary conditions, the buckling load of the composite plate with one free edge and the other rotationally restrained was derived in their further research. Their work verified that the 1st order variational theorem of potential energy is very effective for solving critical buckling load of orthotropic plates with elastically-supported boundaries, and comparison with existing formulas and with the finite element computations leads to an excellent agreement. A comparison of finite element model with the analytical method is represented in literature, in which the local buckling load of a stiffened structure is analysed. The relative error of approximzation model lies between 0.62% and 3.0%, which demonstrates the feasibility of the analytical method for design purpose.

Buckling behaviors of composite structures are sensitive to the layout and shape of components, and the deformation mode might be transformed during the optimized iteration process. Finite element analysis could not usually guarantee the continuity of the optimization model, and the non-gradient based algorithms might be needed to solve the problem, which requires large amount of computing resources. One advantage of the analytical method is that it could provide closed-form prediction models for various buckling modes, and therefore, the follow-up optimization could be processed with high efficiency using the gradient based methods, which is more suitable for the evaluation of overall structural performance.

Concluding Remarks

Commonly used buckling analysis methods for composite structures are discussed, and the potential challenges of these methods in accurate buckling and

post-buckling analysis are critically reviewed. Anisotropic characteristics of composites could affect buckling behaviors, which makes the problem more complicated than metallic materials of isotropic nature. Composites have some special failure mode due to buckling deformation, such as delamination, which might affect buckling resistance of the structure, and analysis algorithms considering some detailed imperfections need further investigations. The prediction of the critical buckling load could be regarded as an eigenvalue problem by solving characteristic system of equations. Energy variational theory has been proved effective by a number of approximation models. In contrast, the iteration process might be needed to estimate post-buckling behavior to trace the equilibrium path. Semi-analytical method is another effective approach to analyze buckling and post-buckling behavior of thin-walled composite structures, which produces fewer degrees of freedom than the finite element models, and it could estimate the critical load with limited computing resources. The critical buckling load of a panel under elastically-constrained boundaries could be predicted by considering the boundary support as a part of the contribution in the potential energy of the system. Thus, more complex structures such as stiffened panels could be analyzed by the discrete plate methods. Buckling mode is susceptible to the change of structural layout, and sometimes the structural optimization model considering the buckling behavior as one constraint might have some continuity problems. By employing the analytical approximation, the solution performance of gradient-based optimization algorithms could be improved. Analytical methods could provide higher computing efficiency and are more suitable for overall demonstrations, while approximation models are usually based on some ideal assumptions. Therefore, more versatile algorithms considering complicated engineering structures might need further research.

REFERENCES

Athiannan K, Palaninathan R. “Buckling of Cylindrical Shells Under Transverse Shear.” Thin-walled Structures 42 (2004): 1307-1328.

Atevens K. A, Ricci R, Davies G. A. O. “Buckling and Post-Buckling of Composite Structures.” Composites 26 (1995): 189-199.

Belytschko T, Lu Y. Y, Gu L. "Element-free Galerkin Methods." International Journal for Numerical Methods in Engineering 37(1994): 229-256.

Bisagni C, Vescovini R. “Analytical Formulation for Local Buckling and Post-buckling Analysis of Stiffened Laminated Panels.” Thin-Walled Structures 47 (2009): 318-334.

Buermann P, Rolfes R, Tessmer J, et al. “A Semi-Analytical Model for Local Post-buckling Analysis of Stringer and Frame Stiffened Cylindrical Panels.” Thin-Walled Structures 44 (2006): 102-104.

Bui T. Q, Nguyen M. N, Zhang C. H. "Buckling analysis of Reissner–Mindlin plates subjected to in-plane edge loads using a shear-locking-free and meshfree method." Engineering Analysis with Boundary Elements 35(2011): 1038-1053.

Byklum E, Amdahl J. “A Simplified Method for Elastic Large Deflection Analysis of Plates and Stiffened Panels due to Local Buckling.” Thin-Walled Structures 40 (2002): 925-953.

Byklum E, Steen E, Amdahl J. “A Semi-analytical Model for Global Buckling and Post Buckling Analysis of Stiffened Panels.” Thin-Walled Structures 42 (2004): 701-707.

Casafont M, Pastor M, Bonada J, et al. "Linear buckling analysis of perforated steel storage rack columns with the Finite Strip Method." Thin-Walled Structures 61(2012): 71-85.

Chai H, Babcock C. D, Knauss W. G. “One Dimensional Modelling of Failure in Laminated Plates by Delamination Buckling.” International Journal of Solids and Structures 17 (1981): 1069-1083.

Chia C Y. “Geometrically Nonlinear Behavior of Composite Plate: a Review.” Applied Mechanics Reviews 41 (1988): 439-451.

Chiara B, Potito C. “An Experimental Investigation into the Buckling and Post-Buckling of CFRP Shells Under Combined Axial and Torsion Loading.” Composite Structures 60 (2003): 391-402.

Dalaei M, Kerr A D. “Analysis of Clamped Rectangular Orthotropic Plates Subjected to a Uniform Lateral Load.” International Journal of Mechanical Sciences 37 (1995): 527-535.

Dawe D. J, Peshkam V. “A Note on Finite Strip Buckling Analysis of Composite Plate Structures.” Composite Structures 34 (1996): 163-168.

Dawe D. J, Wang S, Lam S. S. E. “Finite Strip Analysis of Imperfection Laminated Plates Under End Shortening and Normal Pressure.” International Journal for Numerical Methods in Engineering 38 (1995): 4193–4205.

Dawe D J, Wang S. “Post-buckling Analysis of Composite Laminated Panels.” AIAA Journal 38 (2000): 2160-2170.

Dawe D J, Wang S. “Post-buckling Analysis of Thin Rectangular Laminated Plates by Spline FSM.” Thin Walled Structures 30 (1998): 159-179.

Degenhardt R, Kling A, Rohwer K, et al. “Design and Analysis of Stiffened Composite Panels Including Post-Buckling and Collapse.” Computers & Structures 86 (2008): 919-929.

Eduard R. “Buckling and Post-Buckling Analysis of Stiffened Panels in Wing Box Structures.” International Journal of Solids and Structures 37 (2000): 6795-6824.

Edwards K. L, Davenport C. “Materials for Rotationally Dynamic Components: Rationale for Higher Performance Rotor-Blade Design.” Materials & Design 27 (2006): 31-35.

Eirik Bykluma, Eivind Steen, Jorgen Amdahl. “A semi-Analytical Model for Global Buckling and Postbuckling Analysis of Stiffened Panels.” Thin-walled Structures 42 (2004): 701-717.

Falzon B. G, Hitchings D. “Capturing Mode-switching in Postbuckling Composite Panels Using a Modifed Explicit Procedure.” Composite Structures 60 (2003): 447-453.

Falzon B. G, Stevens K. A, Davies G. O. “Postbuckling Behaviour of a Blade-stiffened Composite Panel Loaded in Uniaxial Compression.” Composites: Part A 31 (2000): 459-468.

Ferreira A. M, Barbosa J. T. “Buckling Behaviour of Composite shells.” Composite Structures 50 (2000): 93-98.

Fiorenzo A. F, Erasmo C. “Advanced Variable Kinematics Ritz and Galerkin Formulations for Accurate Buckling and Vibration Analysis of Anisotropic Laminated Composite Plates.” Composite Structures 94 (2011): 50-67.

Fung Y. C, Sechler E. E. Thin-Shell Structures. New Jersey: Prentice-Hall Inc, 1974, 45-52.

Ghannadpour S. A. M, Ovesy H. R, Nassirnia M. "Buckling analysis of functionally graded plates under thermal loadings using the finite strip method." Computers & Structures 108(2012): 93-99.

Hilburger M. W, Starnes J. H. “Effects of Imperfections of the Buckling Response of Composite Shells.” Thin-Walled Structures 42 (2004): 369-397.

Iuspa L, Ruocco E. “Optimum Topological Design of Simply Supported Composite Stiffened Panels via Genetic Algorithms.” Computers & Structures 86 (2008): 1718-1737.

Iyengar N. G. R, Chakraborty A. “Study of Interaction Curves for Composite Laminate Subjected to in-plane Uniaxial and Shear Loadings.” Compositte Structures 64 (2004): 307-315.

Jaberzadeh E, Azhari M. “Elastic and Inelastic Local Buckling of Stiffened Plates Subjected to Non-uniform Compression Using the Galerkin Method.” Applied Mathematical Modelling 33 (2009): 1874-1885.

Jármai K, Snyman J A, Farkas J. “Minimum Cost Design of a Welded Orthogonally Stiffened Cylindrical Shell.” Computers & Structures 84 (2006): 787-797.

Kidane S, Li G Q, Helms J, et al. “Buckling Load Analysis of Grid Stiffened Composite Cylinders.” Composites, Part B 34 (2003): 1-9.

Kim S H, Kim C G. “Optimal Design of Composite Stiffened Panel with Cohesive Elements using Micro-genetic Algorithm.” Journal of Composite Materials 42 (2008): 2259-2273.

Lanzi L. “A Numerical and Experimental Investigation on Composite Stiffened Panels into Post-buckling.” Thin-walled Structures 42 (2004): 1645-1664.

Lee K, S, Han S. E, Park T. “A Simple Explicit Arc-length Method Using the Dynamic Relaxation Method with Kinetic Damping.” Computers & Structures 89 (2011): 216-233.

Leissa A W. “A Review of Laminated Composite Plate Buckling.” Applied Mechanics Reviews 40 (1987): 575–591.

Liew K. M, Peng L. X, Kitipornchai S. "Analysis of Symmetrically Laminated Folded Plate Structures Using gthe Meshfree Galerkin Method." Mechanics of Advanced Materials and Structures 16(2009): 69-81.

Liu P. F, Hou S. J, Hu X. Y, et al. “Finite Element Analysis of Postbuckling and Delamination of Composite Laminates Using Virtual Crack Closure Technique.” Composite Structures 93 (2011): 1549-1560.

Lynch C, Murphy A, Price M, et al. "The Computational Post Buckling Analysis of Fuselage Stiffened Panels Loaded in Compression." Thin-walled Structures 42 (2004): 1445-1464.

MacNeal-Schwendler Corporation. MSC.Nastran Numerical Methods User's Guide. USA: MacNeal-Schwendler Corporation, 2001, 130-159.

Melin L. G, Schön J. "Buckling Behaviour and Delamination Growth in Impacted Composite Specimens under Fatigue Load: an Experimental Study." Compposites Science & Technology 61 (2001): 1841-1852.

Murphya A, Pricea M, Lynch C, et al. "The Computational Post Buckling Analysis of Fuselage Stiffened Panels Loaded in Shear." Thin-walled Structures 43 (2005): 1455-1474.

Ni Q, Xie J, Iwamoto M. "Buckling Analysis of Laminated Composite Plates with Arbitrary Edge Conditions." Composite Structures 69 (2005): 209-217.

Noor A. K, Burton W. S. "Different Computational Models Used for High Temperature Multi-layered Composite Plates and Shells are Reviewed by Noor and Burton." Applied Mechanics Reviews 45 (1992): 419-445.

Noor A. K, Starnes J. H, Peters J. M. "Thermomechanical Cuckling of Multilayered Composite Panels with Cutouts." AIAA Journal 32 (1994): 1507-1519.

Oh S H, Kim K S, Kim C G. "An Efficient Postbuckling Analysis Technique for Composite Stiffened Curved Panels." Composite Structures 74 (2006): 361-369.

Orifici A. C, Shah S. A, Herszberg I, et al. "Failure Analysis in Postbuckled Composite T-sections." Compsite Structures 86 (2008): 146-153.

Ovesy H. R, Assaee H. "An Investigation on the Post-buckling Behavior of Symmetric Cross-ply Laminated Plates Using a Semi-energy Finite Strip Approach." Composite Structures 71 (2005): 365-370.

Ovesy H. R, Assaee H. "Buckling Characteristics of Some Composite Stiffened Boxes Under Longitudinal Compression or Bending Using Finite Strip Approach." AIAA 2003-1791, 2003.

Ovesy H. R, GhannadPour S. A. M, Morada G. "Geometric Non-linear Analysis of Composite Laminated Plates with Initial Imperfection Under end Shortening, Using two Versions of Finite Strip Method." Composite Structures 71 (2005): 307-314.

Oversy H. R, Ghannadpour S. A. M, Zia-Dehkordi E. "Buckling analysis of moderately thick composite plates and plate structures using an exact finite strip." Composite Structures 95(2013): 697-704.

Paik J. K, Thayamballi A, K. "Buckling Strength of Steel Plating with Elastically Restrained Edges." Thin-Walled Structures 37 (2000): 27-55.

Pedro B. D, Dinar C. "Post-Buckling Behaviour and Strength of Cold-Formed Steel Lipped Channel Columns Experiencing Distortional/Global Interaction." Computers and Structures 89 (2011): 422-434.

Peng L. X, Kitipornchai S, Liew K. M. "Bending Analysis of Folded Plates by the FSDT Meshless Method." Thin-Walled Structures 44(2006): 1138-1160.

Peretz P. F, Bryan G, Rafael P. "A Moderate Deflection Composite Helicopter Rotor Blade Model with an Improved Cross-Sectional Analysis." International Journal of Solids and Structures 46 (2009): 2186-2200.

Peyman K, Ramin S. "Local Buckling and Mode Switching in the Optimum Design of Stiffened Panels." AIAA Journal 46 (2008): 1542-1548.

Qiao P. Z, Shan L. Y. "Explicit Local Buckling Analysis and Design of Fiber-reinforced Plastic Composite Structural Shapes." Composite Structures 70(2005): 468-483.

Qiao P. Z, Zou G. P. "Local Buckling of Composite Fiber-reinforced Plastic Wide-flange Sections." Journal of Structural Engineering 129 (2003): 125-129.

Qiao P. Z, Zou G. P. "Local Buckling of Elastically Restrained Fiber-reinforced Plastic Plates and its Application to Box Section." Journal of Engineering Mechanics 128 (2002): 1324-1330.

Roger G. G, John G. L, Horst D. S. "A Shifted Block Lanczos Algorithm for Solving Sparse Symmetric Generalized Eigenproblems." Journal of Matrix Analysis and Applications 15 (1994): 1-45.

Sallam S, Simitses G. J. "Delamination Buckling and Growth of Flat, Cross-ply Laminates." Composite Structures 4 (1985): 361-368.

Setoodeh A. R, Karami G. "A Solution for the Vibration and Buckling of Composite Laminates with Elastically Restrained Edges." Composite Structures 60 (2003): 245-

253.

Sheinman I, Frostig Y. “Buckling of Delaminated Composite Beams with Shear Deformation Effect.” Journal of Engineering Mechanics 126 (2000): 1148-1155.

Shen H. S. “Thermal Post-buckling Anslysis of Imperfect Laminated Plates Using a Higher-order Shear Deformation Theory.” International Journal of Non-Linear Mechanics 32 (1997): 1035-1050.

Shen H. S. “Thermomechanical Post-buckling Analysis of Imperfect Laminated Plates Using a Higher-order Shear-deformation Theory.” Computers & Structures 66 (1998): 395-409.

Shokrieh M. M, Rajabpour-Shirazi H, Heidari-Rarani M, et al. "Simulation of mode I delamination propagation in multidirectional composites with R-curve effects using VCCT method." Computational Materials Science 65(2012): 66-73.

Singha M. K, Ramachandra L. S, Bandyopadhyay J. N. “Thermal Postbuckling Analysis of Laminated Composite Plates.” Composite Structures 54 (2001): 453-458.

Singh G, Nair P. S, Rao G. V. “An Accurate Lock-free Finite Element for Buckling Analysis of Laminated Plates Under Combined in-plane Loads.” Forschung im Ingenieurwesen 67 (2002): 139-145.

Soutis C. “Failure of Notched CFRP Laminated Due to Fibre Microbuckling: a Topical Review.” Journal of the Mechanical Behavior of Materials 6 (1996): 309-330.

Stamatelos D. G, Labeas G. N, Tserpes K. I. “Analytical Calculation of Local Buckling and Post-buckling Behavior of Isotropic and Orthotropic Stiffened Panels.” Thin-Walled Structures 49 (2011): 422-430.

Suemasu H. “Postbuckling Behaviors of Composite Panels with Multiple Delaminations.” Journal of Composite Materials 27 (1993): 1077-1096.

Timoshenko S, Woinowsky-Krieger S. Theory of Plates and Shells. 2nd ed. New York: McGraw-Hill Book Company, 1959, 1-3.

Topal U, Uzman ü. “Thermal Buckling Load Optimization of Laminated Composite Plates.” Thin-Walled Structures 46 (2008): 667-675.

Variddhi Ungbhakorn, Pairod Singhatanadgid. “Buckling Analysis of Symmetrically Laminated Composite Plates by the Extended Kantorovich Method.” Composite Structures 73 (2006): 120-128.

Wanga S, Dawe D. J. “Buckling of Composite Shell Structures Using the Spline Finite Strip Method.” Composites Part B: Engineering 30 (1999): 351-364.

Wang S, Zhang Y. “Buckling, Post-buckling and Delamination Propagation in Debonded Composite Laminates: Part 1: Theoretical Development.” Composite Structuers 88 (2009): 121-130.

Wang S, Zhang Y. “Buckling, Post-buckling and Delamination Propagation in Debonded Composite Laminates: Part 2: Numerical Applications.” Composite Structuers 88 (2009): 131-146.

Yuan S, Jin Y. “Computation of Elastic Buckling Loads of Rectangular Thin Plates Using the Extended Kantorovich Method.” Computers & Structures 66 (1998): 861-867.

Zhang J Z. “Stability and Bifurcation of Doubly Curved Shallow Panels under Quasi-static Uniform Load.” International Journal of Non Linear Mechanics 38 (2003): 457-466

Zhang Y, Wang S. “Buckling, Post-buckling and Delamination Propagation in Debonded Composite Laminates, Part 1: Theoretical Development.” Composite Structures 88 (2009): 121-130.

Zhao Q, Ding Y L, Jin H B. “A Layout Optimization Method of Composite Wing Structures based on Carrying Efficiency Criterion.” Chinese Journal of Aeronautics 24 (2011): 425-433.

Zimmermann R, Klein H, Kling A. “Buckling and Post Buckling of Stringer Stiffened Fibre Composite Curved Panels-Tests and Computations.” Composite Structures, 73 (2006): 150-161.

Zimmermann R, Rolfes R. “POSICOSS—Improved Postbuckling Simulation for Design of Fibre Composite stiffened Fuselage Structures.” Composite Structures 73 (2006): 171-174.